Deep learning of normal form autoencoders for universal, parameter-dependent dynamics

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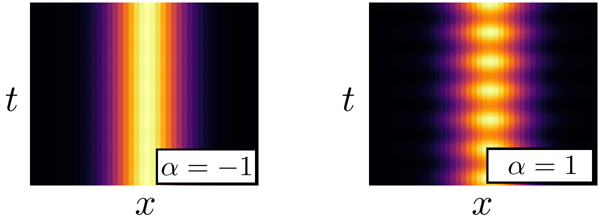
From high to low dimensional models

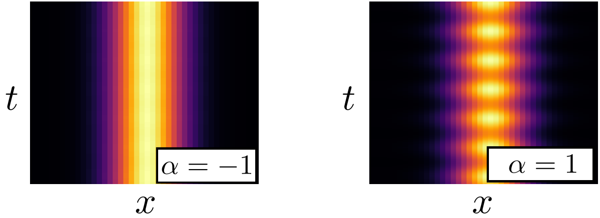
Consider a spatiotemporal system $u(x,t)$, parameterized by α. Different choices of α produce different patterns.



From high to low dimensional models

Consider a spatiotemporal system, parameterized by α. Different choices of α produce different patterns.

Can we construct **underlying low-dimensional models that capture α-dependence?**

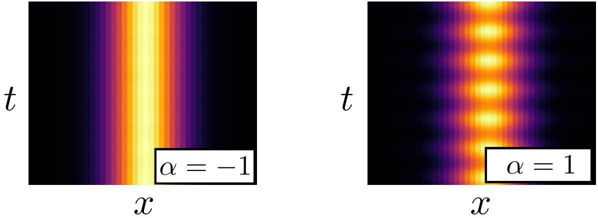


From high to low dimensional models

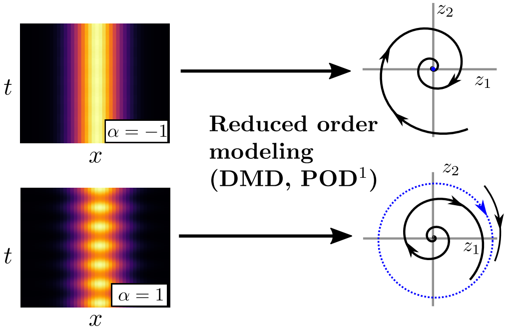
Mathematically, we would like to find Φ, g such that

$$\begin{aligned} \dot{\boldsymbol{u}} &= f(\boldsymbol{u}; \alpha), \quad \text{(Original dynamics)} \\ \dot{z} &= g(z; \beta), \quad \text{(Low dim. model)} \end{aligned}$$

and $(z, \beta) = \Phi(\boldsymbol{u}, \alpha)$.

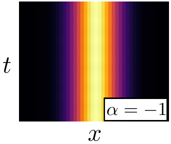


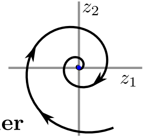
State of the art for data-driven methods

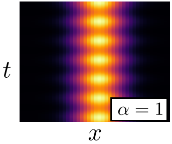


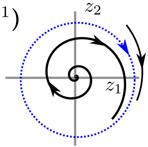
[1] Berkooz et al. *Annual review of fluid mechanics* (1993) ; Tu et al. *Journal of Computational Dynamics* (2014)

State of the art









Reduced order modeling (DMD, POD[1])

Model discovery (SINDy[2])

$\dot{z} = (-1 - i)z$

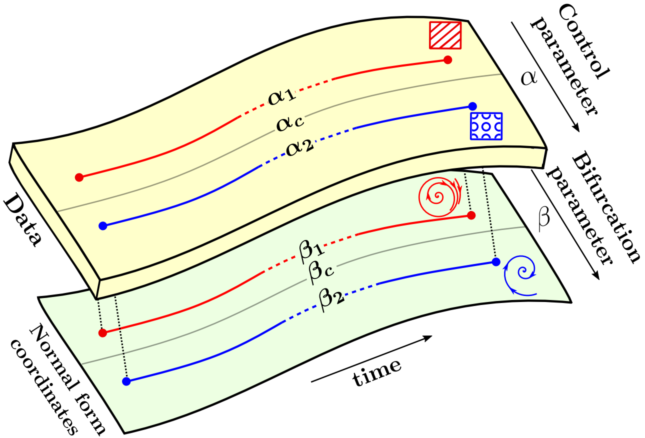
Different experiments, different models

$\dot{z} = (1 - i)z + O(|z|^2)$

[2] Brunton et al. *PNAS* (2016)

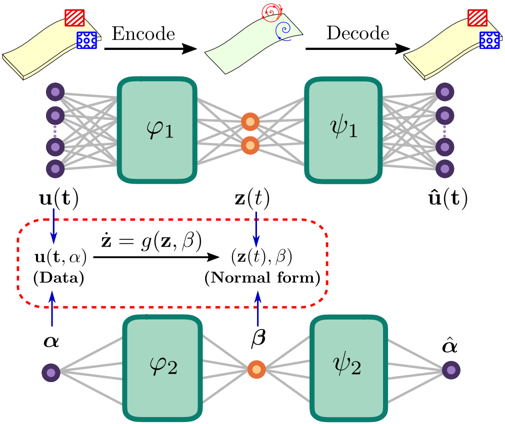
Our approach

Key idea: Use normal forms as reduced order models.



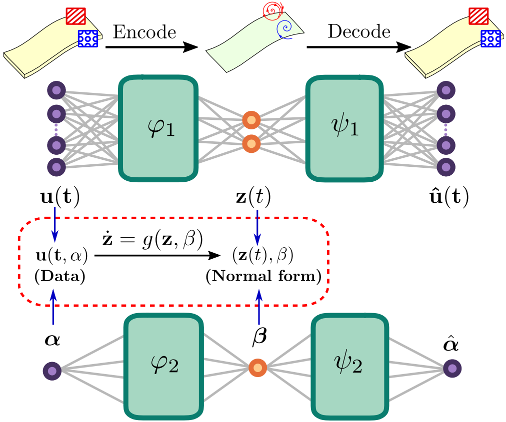
Coupled autoencoders for learning normal form coordinates

- Separate autoencoders for state and parameter.
- Latent space constrained by normal form.
- Existence of non-unique, feasible solutions guaranteed by center manifold theorems.
- $(u, \alpha) \mapsto (z, \beta)$ interpretable as center manifold restriction.
- Trained with simulations $u(t)$ for different α.



Loss function terms

- Simplify!!
- AE loss: $\|u - \psi_1\varphi_1 u\|_2^2 + \|\alpha - \psi_2\varphi_2\alpha\|_2^2$
- Consistency loss terms:
  - $\|(\nabla_u\varphi_1)\dot{u} - g(\varphi_1 u, \varphi_2\alpha)\|_2^2$
  - $\|\dot{u} - (\nabla_z\psi_2)g(\varphi_1 u, \varphi_2\alpha)\|_2^2$
- Orientation loss terms:
  - $\|\text{sgn}(\varphi_2\alpha) - \alpha\|_1$
  - $\|\varphi_1(0)\|_1$ $(0 \mapsto 0)$
  - $\|\mathbb{E}_t u\|$ (Hopf)



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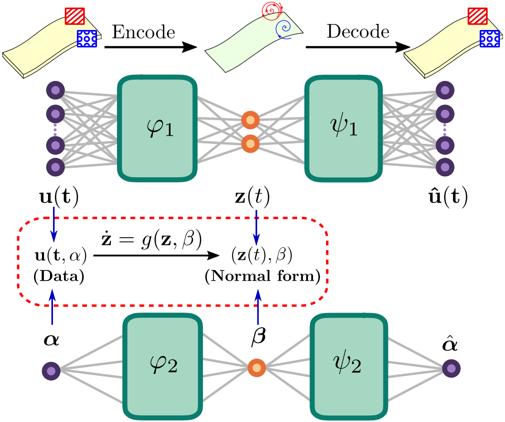

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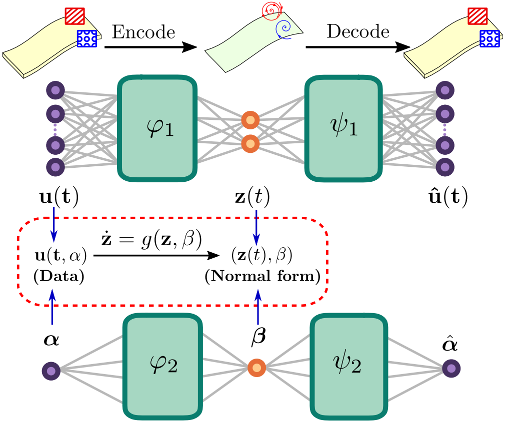
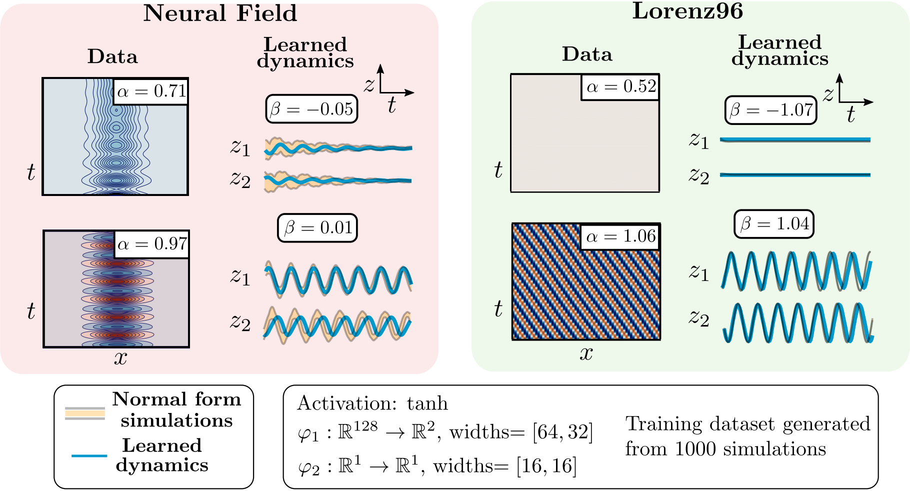


Results: Hopf bifurcations in 1D spatio temporal systems



Activation: tanh

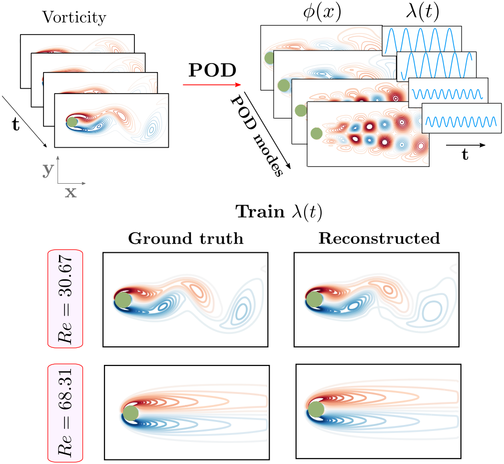
$\varphi_1 : \mathbb{R}^{128} \to \mathbb{R}^2$, widths= $[64, 32]$

$\varphi_2 : \mathbb{R}^1 \to \mathbb{R}^1$, widths= $[16, 16]$

Training dataset generated from 1000 simulations

Treating higher dimensional problems: Navier Stokes

- Fully connected MLP neural networks present *curse of dimensionality*
- POD reduces dimension $\rightarrow$ computationally cheap
- Discover parameterization in low dimensional space $\lambda(t)$
- Reconstruction using 4 modes reconstructs vortices successfully!



Outlook

- Implemented a reduced order modeling framework for parameter-varying data.
- Low-dimensional parameter-dependent models $\rightarrow$ normal forms.
- High to low dimension: learn the restriction to a *center manifold* with neural networks.
- The restriction exists theoretically!
- **Future work:** global bifurcations, use normal forms as building blocks for model discovery.

Relevant work

Champion, K. *et al.* Data-driven discovery of coordinates and governing equations. *Proc. Natl. Acad. Sci. U.S.A.* **116** (2019).

Yair, O. *et al.* Reconstruction of normal forms by learning informed observation geometries from data. *Proc. Natl. Acad. Sci. U.S.A.* **114** (2017).

Brunton, S. L. *et al.* Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *Proc. Natl. Acad. Sci. U.S.A.* **113** (2016).

Tu, J. H. *et al.* On dynamic mode decomposition: Theory and applications. *J. Comput. Dyn.* **1** (2014).

Berkooz, G. *et al.* The proper orthogonal decomposition in the analysis of turbulent flows. *Annu. Rev. Fluid Mech.* **25** (1993).